FORCES CAUSED BY POST-TENSIONING IN CONTINUOUS CONCRETE GIRDERS

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ABSTRACT

The paper is dealing with continuous post-tensioned girders in elastic range. Different load cases are considered: post-tensioning, total load, a single span loaded. The tendon layout is continuous or each span is supplied by an individual cable. The tendon is bonded by subsequent grouting, or unbonded. For the possibility of relatively simple comparison the solution is given analytically in closed formulae. The method gives a way to analyze many other cases.-The application of the force method leads to systems of linear equations the coefficients of which have remarkable properties. In the case of a continuous girder with a continuous tendon the matrix is a bordered uniform tridiagonal matrix. This property enables to calculate the inverse by help of the Chebyshev polynomials of the second kind. In the case of a continuous girder post-tensioned in each span separately the corresponding matrix is a two-by-two block matrix, the blocks of its inverse can be obtained in a relatively simple form. The final results can be expressed by means of hyperbolic and trigonometric functions.

1. INTRODUCTION

At the very beginning of prestressed concrete construction the two basic forms of post-tensioned structures were already known. The classical structures initiated by [6] had internal tendons and the ducts were grouted. The grouting was said to solve two problems: to ensure the bond and the protection against corrosion. Characteristic early examples for the external post-tensioning date back to other early prestressed concrete structures suggested by [3], [5]. The advantages and disadvantages of the bonded and unbonded tendons were widely discussed, e. g. by [2]. That time no significant difference was supposed between the two systems from the point of view of protection against corrosion. At the same time it was emphasized that the load capacity in case of bonded tendon is generally higher than that of otherwise same member but without bond between concrete and prestressing steel, e. g. discussed by [9]..The role of the cable layout (among other questions) was shown in [13]. Recently, it is known that the problem exists in case of continuous slabs. too [4].

To-day it is well known, that the bonded tendons are endangered by corrosion even in the case of a possibly good grouting. The unbonded and protected internal tendons and external cables forge ahead renouncing the better distribution of internal

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forces and a higher load capacity. It is obvious, that the durability of structures is worth of the slightly less favorable development of forces and stresses. However, it is advisable to know more about the different sorts of post-tensioned structures.

2. FORCES IN VARIOUS POST-TENSIONED CONTINUOUS GIRDERS

There are different numerical procedures to calculate the forces in prestressed concrete structures in elastic range. However, an analytical solution, which gives the results in explicit forms, has many advantages. It is shown that the calculation of the forces in statically indeterminate structures contributes to the determination losses due to elastic deformation if there are more subsequently tensioned cables [10].

No doubt, the formulae are received in a relatively sophisticated way. Even therefore, to show the method and to reach the main objective of this investigation, i. e. the difference between members with bonded and unbonded tendons, some restrictions will be made. The structure is a continuous girder over n+1 bays. The span is everywhere the same and the cross section is constant. A single tendon is lead along the whole girder or each span is prestressed by an individual tendon anchored over the supports. The tendon layout is the same in each bay, otherwise it can be arbitrary. The friction is neglected and mentioned shortly in an other point. The other losses are not taken in account, or it can be supposed, that the prestressing force is already reduced. The prestressing force and its horizontal component will be taken to be equal and the axial deformation of the concrete will be neglected. No non-prestressed reinforcement will be taken into consideration. It is supposed that the effect of the ducts is constant along the whole structure. In case of subsequent grouting a perfect bond is predicted. The load is assumed to be the same in each bay or a single bay is loaded. These limitations are for the benefit of the comparison between different cases. Nevertheless there are practical examples, where the conditions almost fully correspond to the given assumptions, e. g. in case of strengthening by post-tensioning, see [11].

To determine the redundant quantities the force method is used. As unknowns the moments over the intermediate supports, and in case of unbonded tendons the increase of the force in them will be taken. The primary structure is chosen like by [14] as well as by [1]. The equations to solve the redundant systems will be written that way that the coefficient matrices of the systems of equations - if needed - could be given as hypermatrices, the inverses of which are easy to handle as suggested by [8]

2.1. Cases of Tensioning, Cable Layout, Grouting and Loading

2.1.1. Continuous Girder with a Continuous Tendon

The general layout of the girder is shown in Figure 1. The system of equations of continuous girders contain the well known shaped coefficient matrix A. The load vector in case of post-tensioning and of total load is to be written by the help of **e**:

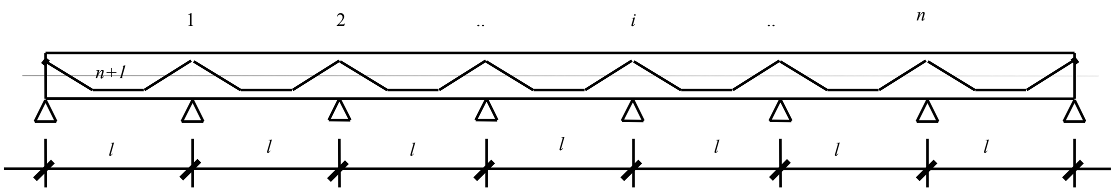

Figure 1 Continuous girder with continuous tendon (schematic sketch)

$$\mathbf{A}=\begin{bmatrix}4&1&&&&\\1&4&1&&&\\&.&.&.&&\\&&.&.&.&\\&&&1&4&1\\&&&&1&4\end{bmatrix},\quad \mathbf{e}=\begin{bmatrix}1\\1\\1\\.\\.\\1\end{bmatrix}.$$

2.1.2. Post-Tensioning – Application of the Force Method in General

In case of post-tensioning, the equation of the force method is the following:

$$\frac{1}{K_I}\mathbf{Ay}=\frac{1}{K_I}2sP\mathbf{e}\quad\Rightarrow\quad\mathbf{y}=2sP\mathbf{A}^{-1}\mathbf{e}\,.$$

2.1.3. Subsequently Grouted Girder Under Total Load

This case is described by the equation:

$$\frac{1}{K_{II}}\mathbf{Ay}=\frac{1}{K_{II}}2tQ\mathbf{e}\quad\Rightarrow\quad\mathbf{y}=2tQ\mathbf{A}^{-1}\mathbf{e}\,.$$

2.1.4. Girder with Unbonded Tendon Under Total Load

In this case the basic equation has the form:

$$\begin{bmatrix}\mathbf{A}&2s\mathbf{e}\\2s\mathbf{e}^T&z\end{bmatrix}\begin{bmatrix}\mathbf{y}\\y_{n+1}\end{bmatrix}=Q\begin{bmatrix}2t\mathbf{e}\\q\end{bmatrix}. \tag{1}$$

The inverse of the bordered matrix is

$$\begin{bmatrix}\mathbf{A}&\mathbf{b}\\\mathbf{b}^T&d\end{bmatrix}^{-1}=\begin{bmatrix}\mathbf{A}^{-1}+\dfrac{\mathbf{A}^{-1}\mathbf{b}\mathbf{b}^T\mathbf{A}^{-1}}{d-\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}}&-\dfrac{\mathbf{A}^{-1}\mathbf{b}}{d-\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}}\\-\dfrac{\mathbf{b}^T\mathbf{A}^{-1}}{d-\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}}&\dfrac{1}{d-\mathbf{b}^T\mathbf{A}^{-1}\mathbf{b}}\end{bmatrix}.$$

Therefore the solution of Eq. (1) will be written as

$$\begin{bmatrix} \mathbf{y} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{-1} + \dfrac{4s^2\mathbf{A}^{-1}\mathbf{e}\mathbf{e}^T\mathbf{A}^{-1}}{z-4s^2\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}} & -\dfrac{2s\mathbf{A}^{-1}}{z-4s^2\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}} \\ -\dfrac{2s\mathbf{e}^T\mathbf{A}^{-1}}{z-4s^2\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}} & \dfrac{1}{z-4s^2\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}} \end{bmatrix} \begin{bmatrix} 2t\mathbf{e} \\ q \end{bmatrix} Q.$$

Performing the equations we get

$$\mathbf{y} = 2Q\frac{tz-sq}{z-4s^2\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}}\mathbf{A}^{-1}\mathbf{e} \quad ; \quad y_{n+1} = Q\frac{q-4st\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}}{z-4s^2\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}}.$$

The expression $\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}$ is the sum of the elements of the inverse $\mathbf{A}^{-1}$. A cumbersome calculation leads to the result:

$$\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e} = \begin{cases} \dfrac{1}{2\cosh\theta+2}\left\{ n + \dfrac{2\sinh\frac{n+1}{2}\theta\cosh\frac{n}{2}\theta}{\cosh\frac{\theta}{2}} \right\}, & \textit{if } n \textit{ is odd}; \\ \dfrac{1}{2\cosh\theta+2}\left\{ n + \dfrac{2\cosh\frac{n+1}{2}\theta\sinh\frac{n}{2}\theta}{\cosh\frac{\theta}{2}} \right\}, & \textit{if } n \textit{ is even}. \end{cases} \tag{2}$$

The elements of the vector $\mathbf{A}^{-1}\mathbf{e}$ - i. e. the sum S_i of the row elements of the inverse matrix - are the following:

$$S_i = \frac{1}{2\cosh\theta+2}\left\{ 1+(-1)^{i-1}\frac{\sinh i\theta+(-1)^{n-1}\sinh(n+1-i)\theta}{\sinh(n+1)\theta} \right\}. \tag{3}$$

For all the above expressions the parameter θ is defined by

$$4 = 2\cosh\theta, \quad \Rightarrow \quad \theta = \ln\left(2+\sqrt{3}\right). \tag{4}$$

So in the expressions (2) and (3)

$$\frac{1}{\cosh\theta+2} = \frac{1}{6}.$$

2.1.5. The Load Acting only in One Span

The definition of the parameter θ is the same as in (4). The solution of the equation for the similar cases of post-tensioning is the following: If a single span is post-tensioned

$$\frac{1}{Q}\mathbf{y} = 2sP\mathbf{A}^{-1}\left(\mathbf{e}_{j-1} + \mathbf{e}_j\right).$$

In case of a load in a single span

$$\frac{1}{Q}\mathbf{y} = 2tQ\mathbf{A}^{-1}\left(\mathbf{e}_{j-1} + \mathbf{e}_j\right).$$

If a single bay of a girder with unbonded tendon is loaded

$$\frac{1}{Q}y_{n+1} = -\frac{2s\mathbf{e}^T\mathbf{A}^{-1}\left(\mathbf{e}_{j-1} - \mathbf{e}_j\right)2t}{z - 4s^2\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}} + \frac{q}{z - 4s^2\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}}.$$

After calculation the following result is received for the unknowns:

$$\mathbf{y} = \frac{2Q}{z - 4s^2\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}}\left\{\left(4s^2t\left(\mathbf{e}^T\mathbf{A}^{-1}\left(\mathbf{e}_{j-1} + \mathbf{e}_j\right) - sq\right)\mathbf{A}^{-1}\mathbf{e}\right) - \left(4s^2t\left(\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}\right) - tz\right)\mathbf{A}^{-1}\left(\mathbf{e}_{j-1} + \mathbf{e}_j\right)\right\},$$

where j=1, 2, ..., n, and $\mathbf{e}_0$ =0. The force in the tendon is

$$\mathbf{y}_{n+1} = Q\frac{q - 4st\left(\mathbf{e}^T\mathbf{A}^{-1}\left(\mathbf{e}_{j-1} + \mathbf{e}_j\right)\right)}{z - 4s^2\left(\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}\right)}$$

The product in this formula is

$$\mathbf{e}^T\mathbf{A}^{-1}\left(\mathbf{e}_{j-1} + \mathbf{e}_j\right) = \frac{1}{2\cosh\theta + 2}\left\{2 + (-1)^{j-1}\frac{1}{\sinh(n+1)\theta}\left(\sinh j\theta - \sinh(j-1)\theta + \sinh(n+1-j)\theta - \sinh(n+2-j)\theta\right)\right\}$$

$$\mathbf{e}^T\mathbf{A}^{-1}\left(\mathbf{e}_{j-1} + \mathbf{e}_j\right) = \frac{1}{2\cosh\theta + 2}\left\{2 + (-1)^{j-1}\frac{1}{\sinh(n+1)\theta}\left(\sinh(n+1-j)\theta - \sinh(n+2-j)\theta - \sinh j\theta + \sinh(j-1)\theta\right)\right\}$$

in the upper formula n is odd, in the lower one it is even.

2.1.6. Continuous Girder Post-Tensioned by Tendons Reaching in each Bay from the Left Side Support to the Right One

The general arrengement of the girder is to be seen in Figure 2. If it is supposed, that all the tendons are tensioned simultaneously. The force method leads to the linear system of equations

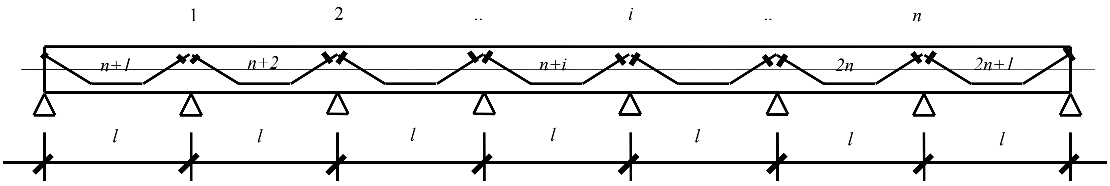

Figure 2 Continuous girder with tendons anchored at the ends of each span

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{y}_P \end{bmatrix} = \mathbf{a}_0 \quad \Rightarrow \quad \begin{bmatrix} \mathbf{y} \\ \mathbf{y}_P \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{D} \end{bmatrix}^{-1} \mathbf{a}_0 . \tag{5}$$

In case of total load and post-tensioning

$$\mathbf{a}_0 = \begin{bmatrix} \mathbf{e}_{(n)} \\ f\mathbf{e}_{(n+1)} \end{bmatrix}, \tag{6}$$

and in case if a single bay is loaded

$$\mathbf{a}_0 = \begin{bmatrix} \mathbf{e}_{j-1} + \mathbf{e}_j \\ 0 \end{bmatrix},$$

where

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{D} \end{bmatrix} = \begin{array}{r} 1) \\ 2) \\ . \\ i) \\ . \\ n) \\ n+1) \\ n+2) \\ . \\ n+i) \\ . \\ 2n) \\ 2n+1) \end{array} \begin{bmatrix} 4 & 1 & & & & & c & c & & & & & \\ 1 & 4 & 1 & & & & & c & c & & & & \\ & . & . & . & & & & & . & . & & & \\ & & 1 & 4 & 1 & & & & & c & c & & \\ & & & . & . & . & & & & & . & . & \\ & & & & 1 & 4 & & & & & & c & c \\ c & & & & & & v & & & & & & \\ c & c & & & & & & . & & & & & \\ & . & . & & & & & & . & & & & \\ & & c & c & & & & & & v & & & \\ & & & . & . & & & & & & . & & \\ & & & & c & c & & & & & & . & \\ & & & & & c & & & & & & & v \end{bmatrix} \quad ; \quad \mathbf{a}_0 = \begin{bmatrix} 1 \\ 1 \\ . \\ 1 \\ . \\ 1 \\ f \\ f \\ . \\ f \\ . \\ f \\ f \end{bmatrix} \quad ; \quad \mathbf{a}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ i-1 \\ i \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

The inverse can be written in the following form:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \left(\mathbf{A}-\mathbf{B}^T\mathbf{D}\mathbf{B}\right)^{-1} & -\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{D}^{-1} \\ \mathbf{D}^{-1}\mathbf{B}\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}\mathbf{B}\right)^{-1} & \mathbf{D}^{-1}+\mathbf{D}^{-1}\mathbf{B}\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{D}^{-1} \end{bmatrix} . \tag{7}$$

Substituting into (5) we get

$$\mathbf{y} = \left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1}\mathbf{e}_n - \left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{D}^{-1}f\mathbf{e}_{(n+1)} \quad ,$$
$$\mathbf{y}_P = -\mathbf{D}^{-1}\mathbf{B}\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1}\mathbf{e}_n + \mathbf{D}^{-1}f\mathbf{e}_{(n+1)} + \mathbf{D}^{-1}\mathbf{B}\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{D}^{-1}f\mathbf{e}_{(n+1)} \quad . \tag{8}$$

Since $\mathbf{B}^T\mathbf{D}^{-1}\mathbf{e}_{(n+1)} = 2\dfrac{c}{v}\mathbf{e}_{(n)}$,

substituting it into (8) the unknown support moments are

$$\mathbf{y} = \left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1}\mathbf{e}_{(n)}\left\{1-\frac{2fc}{v}\right\},$$

and the forces in the tendons due to external load:

$$\mathbf{y}_P = \left(-1+\frac{2fc}{v}\right)\frac{c}{v}\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & . & & \\ & & . & . & \\ & & & . & . \\ & & & & . & 1 \\ & & & & & 1 \end{bmatrix}\left(\mathbf{A}-\mathbf{B}^T\mathbf{A}\mathbf{B}\right)^{-1}\mathbf{e}_{(n)} + \frac{f}{v}\mathbf{e}_{(n+1)}$$

Now let us calculate the elements of the inverse

$$\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1}.$$

Considering that

$$\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1} = \left\{\begin{bmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{bmatrix} - \frac{c^2}{v}\begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & . & . & . & & \\ & & . & . & . & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}\right\}^{-1} =$$

$$
=\begin{bmatrix} 4-2\frac{c^2}{v} & 1-\frac{c^2}{v} & & & & \\ 1-\frac{c^2}{v} & 4-2\frac{c^2}{v} & 1-\frac{c^2}{v} & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & 1-\frac{c^2}{v} & 4-2\frac{c^2}{v} \end{bmatrix}^{-1} = \frac{1}{1-\frac{c^2}{v}} \begin{bmatrix} 2\cosh\theta & 1 & & & & \\ 1 & 2\cosh\theta & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ & & & & 1 & 2\cosh\theta \end{bmatrix}^{-1},
$$

because

$$
\mathbf{B}^T\mathbf{D}\mathbf{B} = \frac{c^2}{v}\begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & 1 \\ & & & & & 1 \end{bmatrix} = \frac{c^2}{v}\begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{bmatrix}.
$$

Introducing the notation $\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1} = \left[r_{ij}\right]$,

the elements r_{ij} can be expressed in function of c^2/v.

Three cases are to be distinguished: $a)\quad 0<\frac{c^2}{v}<1$; $\quad b)\quad \frac{3}{2}<\frac{c^2}{v}$; $\quad c)\quad 1<\frac{c^2}{v}<\frac{3}{2}$:

$$
a)\quad r_{ij} = (\cosh\theta - 1)\begin{cases} \dfrac{\sinh i\theta\,\sinh(n+1-j)\theta}{\sinh\theta\,\sinh(n+1)\theta}(-1)^{i+j}; & if \quad i\le j \\ \dfrac{\sinh j\theta\,\sinh(n+1-i)\theta}{\sinh\theta\,\sinh(n+1)\theta}(-1)^{i+j}; & if \quad i\ge j \end{cases}. \tag{9}
$$

$$
b)\quad r_{ij} = (\cos\theta - 1)\begin{cases} \dfrac{\sin i\theta\,\sin(n+1-j)\theta}{\sin\theta\,\sin(n+1)\theta}(-1)^{i+j}; & if \quad i\le j \\ \dfrac{\sin j\theta\,\sin(n+1-i)\theta}{\sin\theta\,\sin(n+1)\theta}(-1)^{i+j}; & if \quad i\ge j \end{cases} \tag{10}
$$

$$
c)\quad r_{ij} = (\cosh\theta + 1)\begin{cases} \dfrac{\sinh i\theta\,\sinh(n+1-j)\theta}{\sinh\theta\,\sinh(n+1)\theta}; & if \quad i\le j \\ \dfrac{\sinh j\theta\,\sinh(n+1-i)\theta}{\sinh\theta\,\sinh(n+1)\theta}; & if \quad i\ge j \end{cases} \tag{11}
$$

The definition of θ is in these cases

$$a)\quad \cosh\theta = \frac{2-\frac{c^2}{v}}{1-\frac{c^2}{v}} \quad \Rightarrow \quad \frac{1}{1-\frac{c^2}{v}} = \cosh\theta - 1\,;$$

$$b)\quad \cos\theta = \frac{2-\frac{c^2}{v}}{1-\frac{c^2}{v}} \quad \Rightarrow \quad \frac{1}{1-\frac{c^2}{v}} = \cos\theta - 1\,;$$

$$c)\quad \cosh\theta = -\frac{2-\frac{c^2}{v}}{1-\frac{c^2}{v}} \quad \Rightarrow \quad \frac{1}{1-\frac{c^2}{v}} = -\cosh\theta - 1\,.$$

2.1.7. Post-Tensioned Girder under Total Load

The moment $\mathbf{y}$ can be written considering Eq, (5) and (7), substituting Eq. (6), the elements of the vector

$$\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1}\mathbf{e}_{(n)}\,, \quad \Rightarrow \quad \sum_{j=1}^{n} r_{ij} \quad \text{are the following:}$$

$$a)\quad \sum_{j=1}^{n} r_{ij} = \frac{\cosh\theta-1}{2(\cosh\theta+1)}\left\{1+(-1)^{i-1}\frac{\sinh i\theta+(-1)^{n-1}\sinh(n+1-i)\theta}{\sinh(n+1)\theta}\right\};$$

since $\dfrac{\cosh\theta-1}{2(\cosh\theta+1)} = \dfrac{1}{2}\tanh^2\dfrac{\theta}{2}$, we get

$$\sum_{j=1}^{n} r_{ij} = \frac{1}{2}\tanh^2\frac{\theta}{2}\left\{1+(-1)^{i-1}\frac{\sinh i\theta+(-1)^{n-1}\sinh(n+1-i)\theta}{\sinh(n+1)\theta}\right\}.$$

$b)$ Since $\dfrac{\cos\theta-1}{2(\cos\theta+1)} = -1\dfrac{1}{2}\tan^2\dfrac{\theta}{2}$,

$$\sum_{j=1}^{n} r_{ij} = -\frac{1}{2}\tan^2\frac{\theta}{2}\left\{1+(-1)^{i-1}\frac{\sin(n+1-i)\theta+(-1)^{n-1}\sin i\theta}{\sin(n+1)\theta}\right\}.$$

$c)$ In this case $\displaystyle\sum_{j=1}^{n} r_{ij} = -\frac{1}{2}\coth^2\frac{\theta}{2}\left\{1-\frac{\sinh(n+1-i)\theta+\sinh i\theta}{\sinh(n+1)\theta}\right\}.$

Making use of the results the unknown support moments y_i are

$$y_i = \left(1-\frac{2fc}{v}\right)\sum_{j=1}^{n} r_{ij}\,. \quad \text{Introducing the notation} \quad \psi = -\left(1-\frac{2fc}{v}\right)\frac{c}{v}$$

the forces $\mathbf{y}_{\mathbf{p}_i}$ in the tendons are the following:

For i=1

$$a)\quad y_{p_1} = \psi \frac{1}{2}\tanh^2\frac{\theta}{2}\left\{1+\frac{\sinh\theta+(-1)^{n-1}\sinh\theta}{\sinh(n+1)\theta}\right\}+\frac{f}{v};$$

$$b)\quad y_{p_1} = -\psi \frac{1}{2}\tan^2\frac{\theta}{2}\left\{1+\frac{\sin n\theta+(-1)^{n-1}\sin n\theta}{\sin(n+1)\theta}\right\}+\frac{f}{v};$$

$$c)\quad y_{p_1} = \psi \frac{1}{2}\coth^2\frac{\theta}{2}\left\{1-\frac{\sinh\theta+\sinh n\theta}{\sinh(n+1)\theta}\right\}+\frac{f}{v}.$$

For i=2, 3...,n

$$a)\quad y_{p_i} = \psi \frac{1}{2}\tanh^2\frac{\theta}{2}\left\{2+(-1)^{i-1}\frac{\sinh(n+1-i)\theta-\sinh(n+2-i)\theta+(-1)^{n-1}(\sinh i\theta-\sinh(i-1)\theta)}{\sinh(n+1)\theta}\right\}+\frac{f}{v};$$

$$b)\quad y_{p_i} = -\psi \frac{1}{2}\tan^2\frac{\theta}{2}\left\{2+(-1)^{i-1}\frac{\sin(n+1-i)\theta-\sin(n+2-i)\theta+(-1)^{n-1}(\sin i\theta-\sin(i-1)\theta)}{\sin(n+1)\theta}\right\}+\frac{f}{v};$$

$$c)\quad y_{p_i} = \psi \frac{1}{2}\coth^2\frac{\theta}{2}\left\{2-\frac{\sinh(n+1-i)\theta+\sinh(n+2-i)\theta+\sinh i\theta+\sinh(i-1)\theta}{\sinh(n+1)\theta}\right\}+\frac{f}{v}.$$

For i=n+1

$$a)\quad y_{p_{n+1}} = \psi \frac{1}{2}\tanh^2\frac{\theta}{2}\left\{1+(-1)^{n-1}\frac{\sinh\theta+(-1)^{n-1}\sinh n\theta}{\sinh(n+1)\theta}\right\}+\frac{f}{v};$$

$$b)\quad y_{p_{n+1}} = -\psi \frac{1}{2}\tan^2\frac{\theta}{2}\left\{1+(-1)^{n-1}\frac{\sin\theta+(-1)^{n-1}\sin n\theta}{\sin(n+1)\theta}\right\}+\frac{f}{v};$$

$$c)\quad y_{p_{n+1}} = \psi \frac{1}{2}\coth^2\frac{\theta}{2}\left\{1-\frac{\sinh\theta+\sinh n\theta}{\sinh(n+1)\theta}\right\}+\frac{f}{v}.$$

2.1.8 The Girder Loaded in a Single Span

The unknown moment and tendon forces can be written as

$$\begin{bmatrix}\mathbf{y}\\ \mathbf{y}_p\end{bmatrix}=\begin{bmatrix}\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1}\left(\mathbf{e}_{j-1}+\mathbf{e}_j\right)\\ -\mathbf{D}^{-1}\mathbf{B}\left(\mathbf{A}-\mathbf{B}^T\mathbf{D}^{-1}\mathbf{B}\right)^{-1}\left(\mathbf{e}_{j-1}+\mathbf{e}_j\right)\end{bmatrix}.$$

Using Eqs. (9), (10) and (11), the unknown moments are

$$a)\quad y_i=(\cosh\theta-1)\frac{\sinh i\theta}{\sinh\theta\,\sinh(n+1)\theta}(-1)^{i+j}\left\{\sinh(n+1-j)\theta-\sinh(n+2-j)\theta\right\},\quad if\quad i\le j-1;$$

$$y_i=(\cosh\theta-1)\frac{\sinh(n+1-i)\theta}{\sinh\theta\,\sinh(n+1)\theta}(-1)^{i+j}\left\{\sinh j\theta-\sinh(j-1)\theta\right\},\quad if\quad i\ge j.$$

$$b)\quad y_i=(\cos\theta-1)\frac{\sin i\theta}{\sin\theta\,\sin(n+1)\theta}(-1)^{i+j}\left\{\sin(n+1-j)\theta-\sin(n+2-j)\theta\right\},\quad if\quad i\le j-1;$$

$$y_i=(\cos\theta-1)\frac{\sin(n+1-i)\theta}{\sin\theta\,\sin(n+1)\theta}(-1)^{i+j}\left\{\sin j\theta-\sin(j-1)\theta\right\},\quad if\quad i\ge j.$$

$$c)\quad y_i=(\cosh\theta+1)\frac{\sinh i\theta}{\sinh\theta\,\sinh(n+1)\theta}\left\{\sinh(n+1-j)\theta+\sinh(n+2-j)\theta\right\},\quad if\quad i\le j-1;$$

$$y_i=(\cosh\theta+1)\frac{\sinh(n+1-i)\theta}{\sinh\theta\,\sinh(n+1)\theta}\left\{\sinh j\theta+\sinh(j-1)\theta\right\},\quad if\quad i\ge j.$$

The forces in the tendons due to external load at a single span are

$$(\mathbf{y}_P)=-\frac{c}{v}\begin{bmatrix}1&&&&&\\1&1&&&&\\&1&.&&&\\&&.&.&&\\&&&&&\\&&&&1&1\\&&&&&1\end{bmatrix}\mathbf{y},$$

$$a)\quad (y_P)_1=-\frac{c}{v}(\cosh\theta-1)\frac{1}{\sinh(n+1)\theta}(-1)^{j-1}\left\{\sinh(n+1-j)\theta-\sinh(n+2-j)\theta\right\},$$

$$b)\quad (y_P)_1=-\frac{c}{v}(\cos\theta-1)\frac{1}{\sin(n+1)\theta}(-1)^{j-1}\left\{\sin(n+1-j)\theta-\sin(n+2-j)\theta\right\},$$

$$c)\quad (y_P)_1=-\frac{c}{v}(\cosh\theta+1)\frac{1}{\sinh(n+1)\theta}\left\{\sinh(n+1-j)\theta+\sinh(n+2-j)\theta\right\}.$$

At the intermediate bays (i=2, 3, ..., n):

$$a)\quad (y_P)_i=-\frac{c}{v}\frac{(\cosh\theta-1)(-1)^{i+j}}{\sinh\theta\,\sinh(n+1)\theta}\left\{\sinh i\theta-\sinh(i-1)\theta\right\}\left\{\sinh(n+1-j)\theta-\sinh(n+2-j)\theta\right\};\quad if\quad i\le j-1,$$

$$(y_P)_i=-\frac{c}{v}\frac{(\cosh\theta-1)(-1)^{i+j}}{\sinh\theta\,\sinh(n+1)\theta}\left\{\sinh j\theta-\sinh(j-1)\theta\right\}\left\{\sinh(n+1-i)\theta-\sinh(n+2-i)\theta\right\};\quad if\quad i\ge j-1,$$

$$(y_P)_i=-\frac{c}{v}\frac{(\cosh\theta-1)}{\sinh\theta\,\sinh(n+1)\theta}\left\{\left(\sinh i\theta-\sinh(i-1)\theta\right)\sinh(n+1-i)\theta-\sinh(i-1)\theta\left(\sinh(n+1-i)\theta-\sinh(n+2-i)\theta\right)\right\};$$

$$if\quad i=j.$$

$$b)\quad \left(y_P\right)_i = -\frac{c}{v}\frac{(\cos\theta-1)(-1)^{i+j}}{\sin\theta\,\sin(n+1)\theta}\{\sin i\theta-\sin(i-1)\theta\}\{\sin(n+1-j)\theta-\sin(n+2-j)\theta\}\,;\quad if\quad i\le j-1,$$

$$\left(y_P\right)_i = -\frac{c}{v}\frac{(\cos\theta-1)(-1)^{i+j}}{\sin\theta\,\sin(n+1)\theta}\{\sin j\theta-\sin(j-1)\theta\}\{\sin(n+1-i)\theta-\sin(n+2-i)\theta\}\,;\quad if\quad i\ge j-1,$$

$$\left(y_P\right)_i = -\frac{c}{v}\frac{\cos\theta-1}{\sin\theta\,\sin(n+1)\theta}\{(\sin i\theta-\sin(i-1)\theta)\sin(n+1-i)\theta-\sin(i-1)\theta(\sin(n+1-i)\theta-\sin(n+2-i)\theta)\}\,;$$

$$if\quad i=j.$$

$$c)\quad \left(y_P\right)_i = -\frac{c}{v}\frac{(\cosh\theta+1)}{\sinh\theta\,\sinh(n+1)\theta}\{\sinh i\theta+\sinh(i-1)\theta\}\{\sinh(n+1-j)\theta+\sinh(n+2-j)\theta\}\,;\quad if\quad i\le j-1,$$

$$\left(y_P\right)_i = -\frac{c}{v}\frac{(\cosh\theta+1)}{\sinh\theta\,\sinh(n+1)\theta}\{\sinh j\theta+\sinh(j-1)\theta\}\{\sinh(n+1-i)\theta+\sinh(n+2-i)\theta\}\,;\quad if\quad i\ge j-1,$$

$$\left(y_P\right)_i = -\frac{c}{v}\frac{(\cosh\theta+1)}{\sinh\theta\,\sinh(n+1)\theta}\{(\sinh i\theta-\sinh(i-1)\theta)\sinh(n+1-i)\theta+\sinh(i-1)\theta(\sinh(n+1-i)\theta-\sinh(n+2-i)\theta)\}\,;$$

$$if\quad i=j.$$

Finally, the force in the tendon of the last bay due to the load in a single span

$$a)\quad \left(y_P\right)_{n+1} = -\frac{c}{v}(\cosh\theta-1)\frac{1}{\sinh(n+1)\theta}(-1)^{n-j}\{\sinh j\theta-\sinh(j-1)\theta\},$$

$$b)\quad \left(y_P\right)_{n+1} = -\frac{c}{v}(\cos\theta-1)\frac{1}{\sin(n+1)\theta}(-1)^{n-j}\{\sin j\theta-\sin(j-1)\theta\},$$

$$c)\quad \left(y_P\right)_{n+1} = -\frac{c}{v}(\cosh\theta+1)\frac{1}{\sinh(n+1)\theta}\{\sinh j\theta+\sinh(j-1)\theta\}.$$

2.2. Remarks and Statements

The limited extension of this paper does not allow to deal with numerical results in detail. Also it was not possible to refer to a previous work [12], where the friction was considered on a similar way, but from an other point of view. It is but to be seen, that there are large differences, e.g. in the cracking moments of girders with bonded or unbonded tendons, but normally the difference does not exceed 10-12%. It is to be mentioned that the method [7] was developed in [1], [9] for a more exact calculation of post-tensioned members with unbonded tendons, comparison between cases of bonded and unbonded tendons has shown in given cases relative large differences in load capacity. It can be said that there is no reason to offer the better protection against corrosion for the increase of the resistance of the structure. It is but advantageous to have methods for a better estimation of the difference.

3. CONCLUSION

The aim of prestressing is to create an advantageous distribution of internal forces in concrete members. In case of continuous girders, redundant forces are acting due to prestressing. If the primary system is taking fhat way, that the unknowns are moments, their signs are normally opposite to those of moments due to gravitational loads. Applying various post-tensioning, the forces can be calculated aanalytically for special regular arrangements of the girder and the cable. The analytical solution presented in the paper enables to carry out parametric analyses and other studies The applied procedures of the linear algebra enable enable to have a wider overview on the behaviour ot these structures.

SYMBOLS

A, B, D = coefficient matrix, blocks of hypermatrix
b = bordering vector
c = element of block representing the action of tendons
d = corner element of bordered matrix

e = vector with all unit elements

$\mathbf{e}_i$ = vector with the unit element at *i*
$\mathbf{e}_{(n)}$ = vector with *n* unit elements
f = parameter representing the prestress
i, j = row and column indices
n = number of unkown moments
P = intensity of prestressing force
q = parameter of external load

s = reduced load parameter

Q = intensity of external load
r = auxiliary notation
t = reduced laod parameter
$^{\mathrm{T}}$ = symbol of transposed matrix or vector
v = coefficent matrix element for tendons
y, **y** = moments at the supports

y_P, $\mathbf{y}_P$ = forces in the tendons
z = corner element
ϕ, θ = parameter
Ψ = auxiliary notation
K^{I}, K^{II} = stiffness considering the ducts and grouting respectively

The quantities are reduced to a dimensionless form. To have the real results MN and m are to be used.

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